



PHYS 102 – General Physics II Midterm Exam 1

Duration: 120 minutes

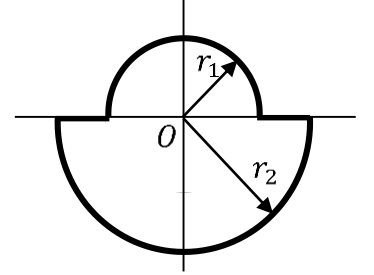
Saturday, 01 March 2025; 09:30

1. A non-conducting wire that carries a uniform line charge of density $+\lambda$ is bent into the shape shown in the figure. The two co-planar semicircles with radius r_1 and r_2 share the same center and are joined by straight-line segments.

(a) (5 Pts.) Find the total charge on the wire.

(b) (10 Pts.) Find the magnitude of the electric field at the origin.

(c) (10 Pts.) Find the electric potential at the origin. (The potential at infinity is zero.)



Solution: (a) Length of the wire is the sum of lengths of the individual pieces.

$$L = \pi r_1 + \pi r_2 + 2(r_2 - r_1), \quad Q = \lambda L = \pi(r_2 + r_1)\lambda + 2(r_2 - r_1)\lambda.$$

(b) Because of the symmetry of the charge distribution, electric fields created by the straight line sections cancel each other. Electric field of a semicircle can be calculated using Coulomb's law. We have

$$E_x = \int dE_x = \frac{-\lambda}{4\pi\epsilon_0 r} \int_0^\pi \cos \theta \, d\theta = 0, \quad E_y = \int dE_y = \frac{-\lambda}{4\pi\epsilon_0 r} \int_0^\pi \sin \theta \, d\theta = \frac{-\lambda}{2\pi\epsilon_0 r}$$

Therefore, magnitude of the electric field created by the two semicircular parts is

$$|\vec{E}| = \frac{\lambda}{2\pi\epsilon_0 r_1} - \frac{\lambda}{2\pi\epsilon_0 r_2} \rightarrow |\vec{E}| = \frac{\lambda(r_2 - r_1)}{2\pi\epsilon_0 r_1 r_2}$$

(c) Since, on the semicircles,

$$dV = \frac{dq}{4\pi\epsilon_0 r} \rightarrow V = \int \frac{\lambda r d\theta}{4\pi\epsilon_0 r} = \frac{\lambda}{4\epsilon_0}$$

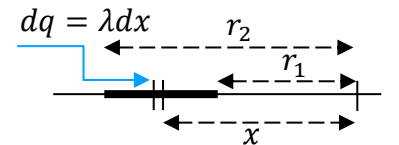
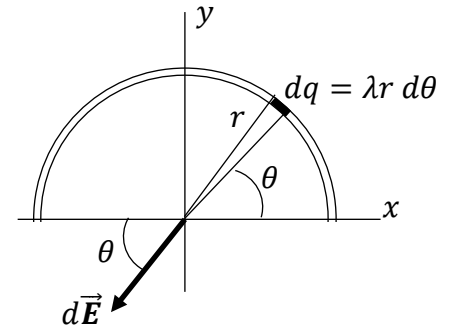
electric potential due to the two circular parts is $V = \lambda/2\epsilon_0$.

For a straight line segment, we have

$$V = \int_{r_1}^{r_2} \frac{\lambda dx}{4\pi\epsilon_0 x} = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right).$$

Therefore, the electric potential at the origin due to two semicircles and two straight line segments is

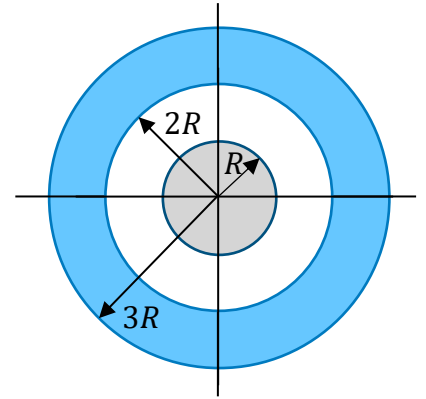
$$V = \frac{\lambda}{2\epsilon_0} + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right).$$



2. A conducting sphere of radius R has total charge Q . The conducting sphere is surrounded by an insulating concentric spherical shell of inner radius $2R$ and outer radius $3R$. The shell has total charge $-Q$ which is distributed uniformly throughout its volume. A cross section of the arrangement is shown in the figure.

Find the magnitude of the electric field as a function of r , where r is the distance from the center, for: (a) (5 Pts.) $r < R$; (b) (5 Pts.) $R < r < 2R$; (c) (10 Pts.) $2R < r < 3R$; (d) (5 Pts.) $3R < r$.

Solution: Consider a spherical Gaussian surface with radius r centered at the origin. Due to symmetry, the electric field will be aligned with the normal $\hat{n}$ to the surface and will have a uniform magnitude across the entire surface. Hence, applying Gauss's law, we have



$$\oint \vec{E} \cdot \hat{n} dA = \frac{Q_{\text{enc}}}{\epsilon_0} \rightarrow E(r)(4\pi r^2) = \frac{Q_{\text{enc}}}{\epsilon_0} \rightarrow E(r) = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2}.$$

(a) The region $0 < r < R$ is conducting. Therefore, $E(r) = 0$, $0 < r < R$.

(b) In the region $R < r < 2R$, we have $Q_{\text{enc}} = Q$. Therefore,

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}, \quad R < r < 2R.$$

(c) In the region $2R < r < 3R$, a total charge $-Q$ is distributed uniformly. The volume charge density in this region is

$$\rho = -\frac{Q}{V} \rightarrow \rho = \frac{-Q}{\frac{4}{3}\pi(3R)^3 - \frac{4}{3}\pi(2R)^3} \rightarrow \rho = \frac{-3Q}{76\pi R^3}.$$

Therefore,

$$Q_{\text{enc}} = Q + \rho \left[\frac{4}{3}\pi r^3 - \frac{4}{3}\pi(2R)^3 \right] \rightarrow Q_{\text{enc}} = Q \left(1 - \frac{r^3 - 8R^3}{19R^3} \right) = Q \left(\frac{27R^3 - r^3}{19R^3} \right)$$

and

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \left(\frac{27R^3 - r^3}{19R^3} \right), \quad 2R < r < 3R.$$

(d) In the region $r > 3R$, we have $Q_{\text{enc}} = 0$. Therefore,

$$E(r) = 0, \quad r > 3R.$$

3. Two identical q charges are placed at $x = 0, y = \pm L$. A third charge Q is placed at $x = \sqrt{2}L, y = 0$. Assume that $q > 0, Q > 0$.

(a) (5 Pts.) Find the force (vector) on charge Q .

(b) (10 Pts.) Find the total electrostatic potential energy of this charge configuration.

(c) (10 Pts.) For which maximum value of Q is there a point on the x -axis that has zero electric field which is not at infinity?

Solution: (a) The two q charges are symmetrically placed on the y -axis. Therefore, the force on the charge Q will be in the $+x$ direction.

$$F_Q = 2 \frac{1}{4\pi\epsilon_0} \frac{qQ}{(\sqrt{3}L)^2} \cos \theta, \quad \cos \theta = \sqrt{\frac{2}{3}} \rightarrow \vec{F}_Q = \frac{qQ}{3\sqrt{6}\pi\epsilon_0 L^2} \hat{i}$$

(b)

$$U = \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2L)} + 2 \frac{1}{4\pi\epsilon_0} \frac{qQ}{\sqrt{3}L} \rightarrow U = \frac{q}{4\pi\epsilon_0 L} \left(\frac{q}{2} + \frac{2Q}{\sqrt{3}} \right).$$

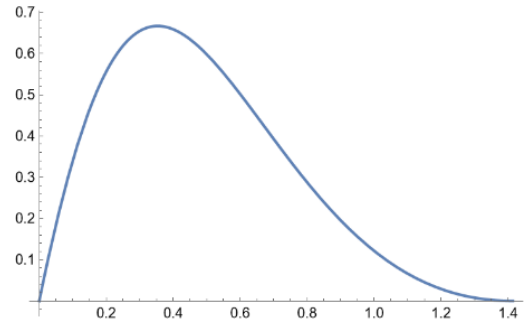
(c) Since all charges are positive, the electric field on the finite x -axis can be zero only in the region $0 < x < \sqrt{2}L$. Because of the symmetry of the charge distribution, the electric field on the x -axis will only have an x component, whose magnitude is found as

$$E(x) = 2 \frac{1}{4\pi\epsilon_0} \frac{q}{(x^2 + L^2)} \cos \phi - \frac{1}{4\pi\epsilon_0} \frac{Q}{(\sqrt{2}L - x)^2}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + L^2}},$$

$$E(x) = \frac{1}{4\pi\epsilon_0} \left[\frac{2qx}{(x^2 + L^2)^{3/2}} - \frac{Q}{(\sqrt{2}L - x)^2} \right].$$

Solving the equation $E(x) = 0$ for Q/q , we get

$$\frac{Q}{q} = \frac{2x(\sqrt{2}L - x)^2}{(x^2 + L^2)^{3/2}} \rightarrow f(u) = \frac{2u(\sqrt{2} - u)^2}{(u^2 + 1)^{3/2}}, \quad \text{where } u = \frac{x}{L}.$$

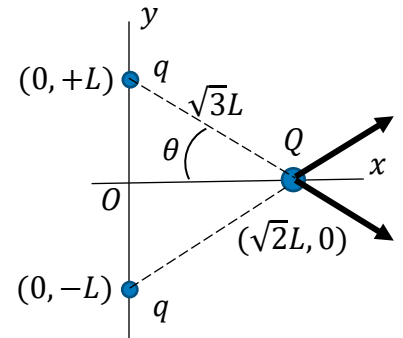
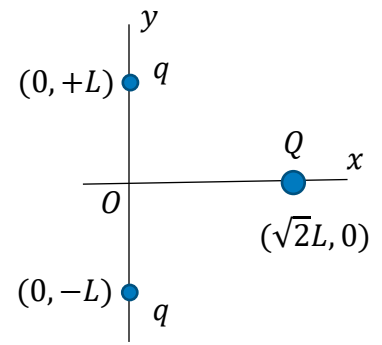


The function $f(u)$ is plotted in the figure. We see that it has a maximum in the region $0 < u < \sqrt{2}$. To find the maximum value of $f(u)$ we solve the equation $f'(u) = 0$.

$$f'(u) = \frac{(2\sqrt{2}u - 1)(u^2 - 2)}{(u^2 + 1)^{5/2}} = 0 \rightarrow u = \pm\sqrt{2}, \quad u = \frac{1}{2\sqrt{2}}$$

meaning that $x = \pm\sqrt{2}L$, and $x = 1/2\sqrt{2}L$. The roots $x = \pm\sqrt{2}L$ can not be physically valid because the electric field can not be zero on the negative x -axis, and there is a point charge at $x = +\sqrt{2}L$. So, the solution is

$$f\left(\frac{1}{2\sqrt{2}}\right) = \frac{2}{3} \rightarrow Q = \frac{2}{3}q.$$



4. Two identical capacitors are made out of square plates of side L , separated by a distance d . The initially uncharged capacitors are connected in series and then connected to a voltage source V_0 .

(a) (5 Pts.) What is the capacitance C of each capacitor?

(b) (5 Pts.) Find the energy stored in one of the capacitors in terms of C .

A dielectric material of constant K is slowly inserted **halfway** into the first capacitor, the voltage source remains connected during this insertion.

(c) (5 Pts.) Find the final charges on both capacitors in terms of C .

(d) (5 Pts.) Find the energy stored in the empty capacitor in terms of C .

(e) (5 Pts.) Find the total work done by the battery during the insertion.

Solution: (a) $C = \epsilon_0 L^2 / d$

(b) The voltage on each capacitor is $V_0/2$. Therefore,

$$U = \frac{1}{2} C \left(\frac{V_0}{2} \right)^2 \rightarrow U = \frac{C V_0^2}{8}.$$

(c)

$$C' = \kappa \frac{C}{2} + \frac{C}{2} \rightarrow C' = \left(\frac{\kappa + 1}{2} \right) C$$

The capacitors are connected in series. Therefore,

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C'} + \frac{1}{C} \rightarrow C_{\text{eq}} = \left(\frac{\kappa + 1}{\kappa + 3} \right) C.$$

The charge on each capacitor is the same, and is equal to

$$Q = C_{\text{eq}} V_0 \rightarrow Q = \left(\frac{\kappa + 1}{\kappa + 3} \right) C V_0.$$

(d)

$$U = \frac{Q^2}{2C} \rightarrow U = \frac{1}{2} \left(\frac{\kappa + 1}{\kappa + 3} \right)^2 C V_0^2.$$

(e) Since the voltage source remains connected during the insertion, charge is transferred. Work done is equal to the amount of charge transferred multiplied by the potential difference. So,

$$W = V_0 \Delta Q = V_0 \left[\left(\frac{\kappa + 1}{\kappa + 3} \right) C V_0 - \frac{1}{2} C V_0 \right] \rightarrow W = \frac{1}{2} \left(\frac{\kappa - 1}{\kappa + 3} \right) C V_0^2.$$

